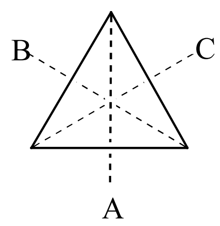
1. (a)



D: cw rotation by 2/3
F: ccw rotation by 2/3

	E	**A**	**B**	**C**	**D**	**F**
E	E	A	B	C	D	F
A	A	E	D	F	B	C
B	B	F	E	D	C	A
C	C	D	F	E	A	B
D	D	C	A	B	F	E
F	F	B	C	A	E	D

Each operation appears exactly once in each row and column in accord with the rearrangement theorem.

(b) {E} is a subgroup by itself.

{E,A}, {E, B}, and {E, C} are subgroups:

	E	**A**
E	E	A
A	A	E

	E	**B**
E	E	B
B	B	E

	E	**C**
E	E	C
C	C	E

{E, D, F} is a subgroup:

	E	**D**	**F**
E	E	D	F
D	D	F	E
F	F	E	D

(c) Using the group multiplication table to calculate similarity transforms, we can show that there are three classes of operations:

Class 1: E
Class 2: A,B,C
Class 3: D,F

2. (a)

	M_1	M_2	M_3	M_4	M_5	M_6
M_1	M_1	M_2	M_3	M_4	M_5	M_6
M_2	M_2	M_1	M_5	M_6	M_3	M_4
M_3	M_3	M_6	M_1	M_5	M_4	M_2
M_4	M_4	M_5	M_6	M_1	M_2	M_3
M_5	M_5	M_4	M_2	M_3	M_6	M_1
M_6	M_6	M_3	M_4	M_2	M_1	M_5

These matrices comprise a group because the requirements of closure, the existence of an identity, and the existence of an inverse for each matrix are all satisfied.

(b) Using the group multiplication table to calculate the similarity transforms, we can show that there are three classes of operations:

Class 1: M_1
Class 2: M_2, M_3, M_4
Class 3: M_5, M_6

(c) Class 1: Trace = 2
Class 2: Trace = 0
Class 3: Trace = -1

All matrices in the same class have the same trace.

(d) Inspection of the group multiplication tables in parts 1(a) and 2(a) shows that the groups are isomorphic with:

$E \longrightarrow M_1$
$A \longrightarrow M_2$
$B \longrightarrow M_3$
$C \longrightarrow M_4$
$D \longrightarrow M_5$
$F \longrightarrow M_6$

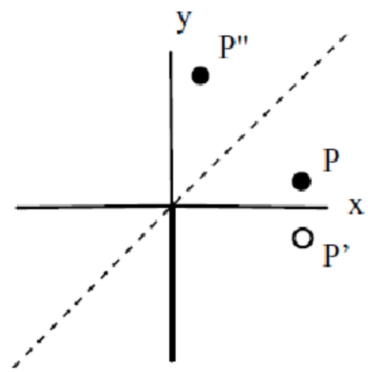
3.

	E	A	B	C	D
E	E	A	B	C	D
A	A	B	C	D	E
B	B	C	D	E	A
C	C	D	E	A	B
D	D	E	A	B	C

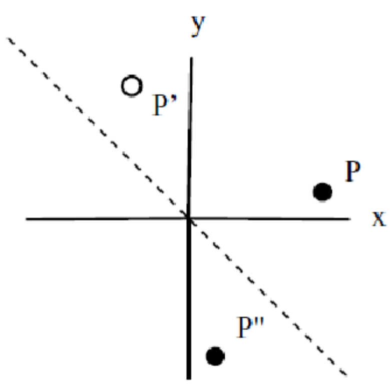
4. (a) Closure requires that the product of any two operations within a group also be in the group. New operations generated are:

- $S_4(z) \bullet S_4(z) \rightarrow \mathbf{C_2(z)}$
- $S_4(z) \bullet C_2(z) \rightarrow \mathbf{S_4^3(z)}$
- $S_4(z) \bullet S_4^3(z) \rightarrow \mathbf{E}$
- $C_2(x) \bullet C_2(x) \rightarrow \mathbf{E}$
- $S_4(z) \bullet C_2(x) \rightarrow \boldsymbol{\sigma}\mathbf{(x=y)}$

$C_2(x)\ (a,b,c) \rightarrow (a,-b,-c)$
$S_4(z)\ (a,-b,-c) \rightarrow (b,a,c)$
$= \sigma(x=y)\ (a,b,c)$



- $C_2(x) \bullet S_4(z) \rightarrow \boldsymbol{\sigma}\mathbf{(x=-y)}$



- $C_2(x) \bullet C_2(z) \rightarrow \mathbf{C_2(y)}$

We can verify that we have all of the operations of the group by using a group multiplication table:

	E	**$C_2(x)$**	**$C_2(y)$**	**$C_2(z)$**	**$S_4(z)$**	**$S_4^3(z)$**	**$\sigma(x=y)$**	**$\sigma(x=-y)$**
E	E	$C_2(x)$	$C_2(y)$	$C_2(z)$	$S_4(z)$	$S_4^3(z)$	$\sigma(x=y)$	$\sigma(x=-y)$
$C_2(x)$	$C_2(x)$	E	$C_2(z)$	$C_2(y)$	$\sigma(x=-y)$	$\sigma(x=y)$	$S_4^3(z)$	$S_4(z)$
$C_2(y)$	$C_2(y)$	$C_2(z)$	E	$C_2(x)$	$\sigma(x=y)$	$\sigma(x=-y)$	$S_4(z)$	$S_4^3(z)$
$C_2(z)$	$C_2(z)$	$C_2(y)$	$C_2(x)$	E	$S_4^3(z)$	$S_4(z)$	$\sigma(x=-y)$	$\sigma(x=y)$
$S_4(z)$	$S_4(z)$	$\sigma(x=y)$	$\sigma(x=-y)$	$S_4^3(z)$	$C_2(z)$	E	$C_2(y)$	$C_2(x)$
$S_4^3(z)$	$S_4^3(z)$	$\sigma(x=-y)$	$\sigma(x=y)$	$S_4(z)$	E	$C_2(z)$	$C_2(x)$	$C_2(y)$
$\sigma(x=y)$	$\sigma(x=y)$	$S_4(z)$	$S_4^3(z)$	$\sigma(x=-y)$	$C_2(x)$	$C_2(y)$	E	$C_2(z)$
$\sigma(x=-y)$	$\sigma(x=-y)$	$S_4^3(z)$	$S_4(z)$	$\sigma(x=y)$	$C_2(y)$	$C_2(x)$	$C_2(z)$	E

We can see that the group is closed, its order=8, and that the rearrangement theorem is satisfied.

(b) From the group multiplication table, we can determine the class structure as in problems 1 and 2:

E

$C_2(z)$

$C_2(x)$, $C_2(y)$

$S_4(z)$, $S_4^3(z)$

$\sigma(x=y)$, $\sigma(x=-y)$

(c) The matrices follow from the transformations under each operation, e.g.

$C_2(z)$ (a,b,c) → (-a,-b,c)

$$\underset{E}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}\quad \underset{C_2(z)}{\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}\quad \underset{C_2(x)}{\begin{bmatrix}1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}}\quad \underset{C_2(y)}{\begin{bmatrix}-1&0&0\\0&1&0\\0&0&-1\end{bmatrix}}\quad \underset{S_4(z)}{\begin{bmatrix}0&-1&0\\1&0&0\\0&0&-1\end{bmatrix}}\quad \underset{S_4^3(z)}{\begin{bmatrix}0&1&0\\-1&0&0\\0&0&-1\end{bmatrix}}$$

$$\underset{\sigma(x=y)}{\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}}\quad \underset{\sigma(x=-y)}{\begin{bmatrix}0&-1&0\\-1&0&0\\0&0&1\end{bmatrix}}$$

(d) We see that each matrix blocks as a 2 x 2 and a 1 x 1:

$$\left[\begin{array}{c|c} 2 \times 2 & 0 \\ \hline 0 & 1 \times 1 \end{array}\right]$$

Thus, this is a reducible representation.

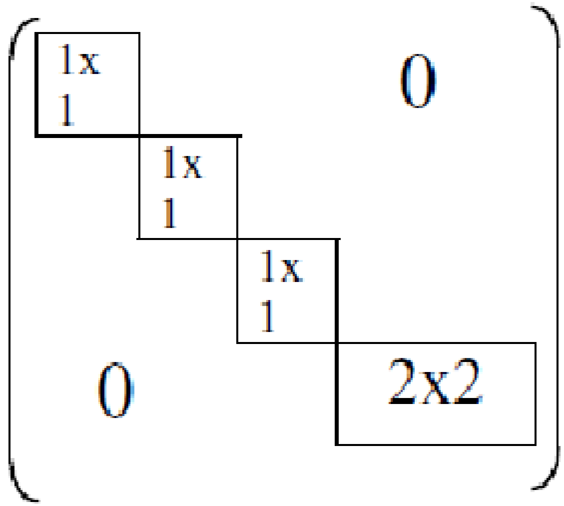
(e) To construct matrices, we can use the results from part (c). For example, consider **C_2(z)** (x,y,z): x → -x; y → -y; z → z

Therefore, $z^2 \rightarrow z^2$; $x^2\text{-}y^2 \rightarrow (-x)^2 - (-y)^2 = x^2\text{-}y^2$; $xy \rightarrow (-x)(-y) = xy$; $xz \rightarrow (-x)(z) = -xz$; and $yz \rightarrow (-y)(z) = -yz$. Thus, the matrices are:

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \quad \underset{C_2(z)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}} \quad \underset{C_2(x)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}} \quad \underset{C_2(y)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}$$

$$\underset{S_4(z)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}} \quad \underset{S_4^3(z)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\sigma(x = y)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}} \quad \underset{\sigma(x = -y)}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}}$$

(f) Each of the matrices from part (e) blocks as three 1 x 1 and one 2 x 2 matrix:



Therefore, these matrices comprise a reducible representation.

(g) Let us assume that the 2 x 2 blocks in parts (c) and (e) are irreducible. We can gather irreducible representations from those matrices:

(c)	E	$C_2(z)$	$C_2(x)$	$C_2(y)$	$S_4(z)$	$S_4^3(z)$	$\sigma(x=y)$	$\sigma(x=-y)$
Γ^1	1	1	-1	-1	-1	-1	1	1
Γ^2	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$
(e)								
Γ^3	1	1	1	1	1	1	1	1
Γ^4	1	1	1	1	-1	-1	-1	-1
Γ^5	1	1	-1	-1	-1	-1	1	1
			(Γ^5 same as Γ^1)					
Γ^6	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

Note that the characters for Γ^6 are the same as those for Γ^2. This suggest that the two are related. In fact, consider the matrix:

$$\underline{\underline{S}} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

which is its own inverse, i.e. $S^{-1} = S$.

You can show that for $\Gamma^6(R) = S^{-1}\,\Gamma^2(R)S$ for all R. Because Γ^6 and Γ^2 are related by a similarity transform, they are **not** unique irreducible representations.

By inspection, the *Great Orthogonality Theorem* holds for the 1-dimensional irreps Γ^1, Γ^3, and Γ^4:

$$\Gamma^1 \bullet \Gamma^1 = \Gamma^3 \bullet \Gamma^3 = \Gamma^4 \bullet \Gamma^4 = 8$$
$$\Gamma^1 \bullet \Gamma^3 = \Gamma^1 \bullet \Gamma^4 = \Gamma^3 \bullet \Gamma^4 = 0$$

$$\sum_R \Gamma^i(R)\Gamma^j(R) = 8\delta_{ij} \; (i, j = 1, 3, or\ 4)$$

For Γ^2 we see by inspection:

$$\sum_R \Gamma^2_{mn}(R)\Gamma^2_{m'n}(R) = \frac{8}{2}\delta_{mm'}\delta_{nn'} \; (m, n = 1\ or\ 2)$$

Furthermore, each set of Γ^2_{mn} (R) is orthogonal to the 1-dimensional irreps, e.g.

$$\sum_R \Gamma^1(R)\Gamma^2_{11}(R) = 0$$

Thus, the representations satisfy the *Great Orthogonality Theorem*.

(h) To summarize the irreps created thus far, listing characters and classes:

E	**C_2**	**$2C_2$'**	**$2S_4$**	**2σ**
1	1	1	1	1
1	1	1	-1	-1
1	1	-1	-1	1
2	-2	0	0	0

We see that there are 5 classes but only 4 irreps. Thus, there should be on more irrep.
Using the *Great Orthogonality Theorem*:
$\sum_i [X^i(E)]^2 = h$; for the above table: $\sum_i [X^i(E)]^2 = 7$ so the missing irrep must have $l = 1$

By inspection, the missing irrep is:

E	**C_2**	**$2C_2$'**	**$2S_4$**	**2σ**
1	1	-1	1	-1